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REFINEMENT OF THE PLASTIC-ZONE BOUNDARY IN THE VICINITY OF A CRACK TIP FOR THE QUASIVISCOUS AND VISCOUS TYPES OF FRACTURE

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A refined solution of the elastoplastic problem of an insulated mode I crack in a thin plate of reasonably large dimensions is obtained. Estimates of the plastic zone in the vicinity of the crack tip are given for quasiviscous and viscous types of fracture.

Key words: *crack, plastic zone, quasibrittle, quasiviscous fracture, viscous fracture.*

1. Formulation of the Problem. Let an infinite thin plate with an insulated crack be loaded by remote tensile constant stresses σ_∞ symmetric about the x axis. The crack is modeled by a bilateral cut of length $2l_0$ [1]. From the viewpoint of applied mechanics, it is reasonable to describe the prefracture zone ahead of the crack tip by two geometrical parameters: the length Δ and width h of this zone [2]. In [2, 3], the types of fracture are classified according to the length of the prefracture zone Δ to the crack length l_0 as follows: brittle fracture ($\Delta = 0$), quasibrittle fracture $\Delta/l_0 = o(1)]$, quasiviscous fracture $[\Delta/l_0 = O(1)$ or $l_0/\Delta = O(1)]$, and viscous fracture $[l_0/\Delta = o(1)]$. We identify the prefracture zone with the plastic zone. This problem has an approximate solution, which provides reasonable accuracy for the cases of brittle and quasibrittle fracture [1], where the remote stress is known to be smaller than the yield stress. In this case, by virtue of the inequality $\sigma_\infty \ll \sigma_m$ (σ_m is the yield stress), the formulas for the stresses do not contain the regular term, namely, the stress σ_∞ [1]. However, for the quasiviscous and viscous types of fracture, for which $\Delta \approx l_0$, $\sigma_\infty = O(\sigma_m)$ and $l_0 < \Delta$, $\sigma_\infty \approx \sigma_m$ ($\sigma_\infty < \sigma_m$), respectively, the approximate solution gives poor accuracy. Our aim is to refine the solution of the problem formulated above and estimate the dimension of the plastic zone in the vicinity of the crack tip using this solution.

2. Stresses in a Plate with a Crack. The algorithm for determining the stresses in a thin plate with a crack consists of two stages. In the first stage, the plate without a crack is loaded by remote tensile stresses σ_∞. In the second stage, the crack is modeled by loading the segment $y = 0$, $|x| \leqslant l_0$f by forces equal but opposite to the forces obtained in the first stage, i.e., by compressive forces $-\sigma_\infty$. The desired solution of the problem is the sum of the solutions obtained in the two stages. For the first stage, the solution is given by

$$\sigma_{xx} = \sigma_{xy} = 0, \qquad \sigma_{yy} = \sigma_\infty.$$

In the second stage, the following boundary conditions are formulated at the crack line $y = 0$, $|x| \leqslant l_0$:

$$\sigma_{xx} = \sigma_{xy} = 0, \qquad \sigma_{yy} = -\sigma_\infty.$$

For the second stage, the stress-tensor components are given by [1]

$$\sigma_{xx} = \operatorname{Re} Z_1 - y \operatorname{Im}[Z_1'], \quad \sigma_{yy} = \operatorname{Re} Z_1 + y \operatorname{Im}[Z_1'], \quad \sigma_{xy} = -y \operatorname{Re}[Z_1']. \tag{1}$$

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Here Z_1 is the holomorphic function

$$Z_1 = \frac{\sigma_\infty}{\pi\sqrt{z^2 - l_0^2}} \int\limits_{-l_0}^{l_0} \frac{\sqrt{l_0^2 - \xi^2}}{z - \xi}\, d\xi = \frac{J}{\sqrt{2\pi(z - l_0)}},$$

where

$$J = \frac{\sqrt{2}\sigma_\infty}{\sqrt{\pi(z + l_0)}} \int\limits_{-l_0}^{l_0} \frac{\sqrt{l_0^2 - \xi^2}}{z - \xi}\, d\xi = \frac{\sqrt{2\pi}\sigma_\infty}{\sqrt{z + l_0}}\Big(z - \sqrt{z^2 - l_0^2}\Big).$$

To study the stress field in the vicinity of the crack tip, we introduce the polar coordinates $z - l_0 = \rho\, \mathrm{e}^{i\theta}$, $\rho = \sqrt{(x - l_0)^2 + y^2}$, and $\theta = \arctan\,[y/(x - l_0)]$ for $x > l_0$ and $\theta = \arctan\,[y/(x - l_0)] + \pi$ for $x \leqslant l_0$. For small ρ, we obtain $J = K_{\mathrm{I}} = \sigma_\infty\sqrt{\pi l_0}$, a constant quantity equal to the stress-intensity factor, which depends on the normal-stress distribution and crack length. Let us calculate $\operatorname{Re} Z_1$, $\operatorname{Im} Z_1$, $\operatorname{Re}\,[Z_1']$, $\operatorname{Im}\,[Z_1']$, $\operatorname{Re} J$, $\operatorname{Im} J$, $\operatorname{Re}\,[J']$, and $\operatorname{Im}\,[J']$ for an arbitrary value of ρ:

$$\operatorname{Re} Z_1 = \frac{1}{\sqrt{2\pi\rho}}\Big(\operatorname{Re} J \cos\frac{\theta}{2} + \operatorname{Im} J \sin\frac{\theta}{2}\Big), \qquad \operatorname{Im} Z_1 = \frac{1}{\sqrt{2\pi\rho}}\Big(-\operatorname{Re} J \sin\frac{\theta}{2} + \operatorname{Im} J \cos\frac{\theta}{2}\Big),$$

$$\operatorname{Re}\,[Z_1'] = \frac{1}{\sqrt{2\pi\rho}}\Big\{\operatorname{Re}\,[J'] \cos\frac{\theta}{2} + \operatorname{Im}\,[J'] \sin\frac{\theta}{2} - \frac{1}{2\rho}\Big(\operatorname{Re} J \cos\frac{3\theta}{2} + \operatorname{Im} J \sin\frac{3\theta}{2}\Big)\Big\},$$

$$\operatorname{Im}\,[Z_1'] = \frac{1}{\sqrt{2\pi\rho}}\Big\{-\operatorname{Re}\,[J'] \sin\frac{\theta}{2} + \operatorname{Im}\,[J'] \cos\frac{\theta}{2} + \frac{1}{2\rho}\Big(\operatorname{Re} J \sin\frac{3\theta}{2} - \operatorname{Im} J \cos\frac{3\theta}{2}\Big)\Big\},$$

$$\operatorname{Re} J = \sqrt{2\pi}\sigma_\infty\Big(\frac{\rho_1}{\sqrt{\rho_2}} \cos\Big[\theta_1 - \frac{\theta_2}{2}\Big] - \sqrt{\rho}\cos\frac{\theta}{2}\Big), \tag{2}$$

$$\operatorname{Im} J = \sqrt{2\pi}\sigma_\infty\Big(\frac{\rho_1}{\sqrt{\rho_2}} \sin\Big[\theta_1 - \frac{\theta_2}{2}\Big] - \sqrt{\rho}\sin\frac{\theta}{2}\Big),$$

$$\operatorname{Re}\,[J'] = \sqrt{2\pi}\sigma_\infty\Big(\frac{1}{\sqrt{\rho_2}}\cos\frac{\theta_2}{2} - \frac{\rho_1}{2\sqrt{\rho_2^3}}\cos\Big[\theta_1 - \frac{3\theta_2}{2}\Big] - \frac{1}{2\sqrt{\rho}}\cos\frac{\theta}{2}\Big),$$

$$\operatorname{Im}\,[J'] = \sqrt{2\pi}\sigma_\infty\Big(-\frac{1}{\sqrt{\rho_2}}\sin\frac{\theta_2}{2} - \frac{\rho_1}{2\sqrt{\rho_2^3}}\sin\Big[\theta_1 - \frac{3\theta_2}{2}\Big] + \frac{1}{2\sqrt{\rho}}\sin\frac{\theta}{2}\Big).$$

Here

$$\rho_1 = \sqrt{x^2 + y^2}; \quad \rho_2 = \sqrt{(x + l_0)^2 + y^2}; \quad \theta_1 = \arctan\,[y/x]; \quad \theta_2 = \arctan\,[y/(x + l_0)].$$

Substitution of (2) into (1) yields the following solution for the second stage:

$$\sigma_{xx} = \frac{1}{\sqrt{2\pi\rho}}\Big\{\operatorname{Re} J \cos\frac{\theta}{2}\Big(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\Big) + \operatorname{Im} J \sin\frac{\theta}{2}\Big(1 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\Big)$$
$$+ 2\rho\Big(\operatorname{Re}\,[J']\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} - \operatorname{Im}\,[J']\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}\Big)\Big\},$$

$$\sigma_{yy} = \frac{1}{\sqrt{2\pi\rho}}\Big\{\operatorname{Re} J \cos\frac{\theta}{2}\Big(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\Big) + \operatorname{Im} J \sin\frac{\theta}{2}\Big(1 - \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\Big) \tag{3}$$
$$- 2\rho\Big(\operatorname{Re}\,[J']\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} - \operatorname{Im}\,[J']\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}\Big)\Big\} + \sigma_\infty,$$

$$\sigma_{xy} = \frac{1}{\sqrt{2\pi\rho}}\Big\{\cos\frac{\theta}{2}\sin\frac{\theta}{2}\Big(\operatorname{Re} J\cos\frac{3\theta}{2} + \operatorname{Im} J \sin\frac{3\theta}{2}\Big) - 2\rho\Big(\operatorname{Re}\,[J']\cos^2\frac{\theta}{2}\sin\frac{\theta}{2} + \operatorname{Im}\,[J']\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\Big)\Big\}.$$

Setting $y = 0$ in formulas (2) and (3), for $x \geqslant l_0$ we obtain [4]

$$\sigma_{xx} = \sigma_\infty x/\sqrt{x^2 - l_0^2} - \sigma_\infty, \qquad \sigma_{yy} = \sigma_\infty x/\sqrt{x^2 - l_0^2}, \qquad \sigma_{xy} = 0. \tag{4}$$

We note that the singularity at the crack tip is due to the fact that the analysis is based on the classical theory of elasticity. In real materials, for example, in metals and alloys, plastic strains occur before the stresses become extremely high. As a result, the stresses are limited by the finite quantity σ_m — the yield stress of the material, and a plastic zone is formed ahead the crack tip. The exact configuration and dimensions of the plastic zone are difficult to determine. To estimate the plastic-zone boundary for the plane stresses, we proceed as follows. We write the von Mises yield criterion in the principal axes:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_m^2. \tag{5}$$

Here

$$\sigma_{1,2} = (\sigma_{xx} + \sigma_{yy})/2 \pm \sqrt{((\sigma_{xx} - \sigma_{yy})/2)^2 + \sigma_{xy}^2}, \qquad \sigma_3 = 0. \tag{6}$$

The third principal stress is obtained for the plane-stress conditions. Inserting (6) into (5), we obtain the equality

$$3((\sigma_{xx} - \sigma_{yy})/2)^2 + 3\sigma_{xy}^2 + ((\sigma_{xx} + \sigma_{yy})/2)^2 = \sigma_m^2. \tag{7}$$

Let $\sigma_m/\sigma_\infty = p$. Substitution of formulas (3) into (7) yields

$$\begin{aligned}
&\{(3/4)\sin^2\theta([\operatorname{Im} J]^2 + [\operatorname{Re} J]^2 + 4\rho^2[\operatorname{Re}[J']]^2 + [\operatorname{Im}[J']]^2) \\
&+ 4\rho[\sin\theta(\operatorname{Re} J \operatorname{Im}[J'] - \operatorname{Im} J \operatorname{Re}[J']) - \cos\theta(\operatorname{Re} J \operatorname{Re}[J'] + \operatorname{Im} J \operatorname{Im}[J'])] \\
&+ ([\operatorname{Re} J]^2\cos^2(\theta/2) + [\operatorname{Im} J]^2\sin^2(\theta/2) + \operatorname{Re} J \operatorname{Im} J \sin\theta)\}/(\rho\sigma_\infty^2) \\
&+ \sqrt{2\pi}\{(\operatorname{Re} J\cos(\theta/2) + \operatorname{Im} J\sin(\theta/2)) - (3/2)\sin\theta[-\operatorname{Re} J\sin(3\theta/2) + \operatorname{Im} J\cos(3\theta/2) \\
&+ 2\rho(\operatorname{Re}[J']\sin(\theta/2) - \operatorname{Im}[J']\cos(\theta/2)]\}/(\sqrt{\rho}\sigma_\infty) + 2\pi(1 - p^2) = 0.
\end{aligned} \tag{8}$$

If, in formula (3) defining the normal stress σ_{yy}, we ignore the regular term (the stress σ_∞) and confine ourselves to the asymptotic behavior of the stresses in the vicinity of the crack tip ($\rho \ll l_0$), then, instead of formula (8), we have the following approximate relation [1] for the plane stresses:

$$\rho_p(\theta) \cong [1 + \cos\theta + (3/2)\sin^2\theta]/(4p^2) \tag{9}$$

($\rho_p = \rho/l_0$ is the dimensionless radius vector).

Figure 1 shows the boundaries of the plastic zone defined by formulas (8) and (9) for $p = 8$ ($x_1 = x/l_0$ and $y_1 = y/l_0$). The solid curve refers to the approximate equation (9) and the dotted curve to Eq. (8). In this case, the boundaries of the plastic zones are close enough. For $p < 8$, however, the results obtained using the refined and approximate solutions differ considerably. Figures 2–4 show the curves obtained using formula (8). One can see that for $p > 1$, the plastic zone does not encompass the crack, which agrees with the Leonov–Panasyuk–Dugdale model [2, 5, 6]. For $y = 0$, the longitudinal dimension of the plastic zone ahead of the crack tip can be comparable to the half-length of the crack for $p \cong 1.1$ (see Fig. 3, quasiviscous fracture) and can be equal to five, six or more half-lengths of the crack for $1 < p < 1.1$ (see Fig. 4 for viscous fracture).

Let us estimate the width (h) and length (Δ) of the plastic zone for $x_1 = 1$ and $y_1 = 0$, respectively. We calculate the plastic-zone width h. In this case, we have $\theta = \pi/2$, $\theta_1 = \arctan[y_1]$, $\theta_2 = \arctan[y_1/2]$, $\rho/l_0 = y_1$, $\rho_1/l_0 = \sqrt{y_1^2 + 1}$, and $\rho_2/l_0 = \sqrt{y_1^2 + 4}$. Even if relation (8) is simplified, it is difficult to express y_1 in terms of p. Therefore, we construct an empirical formula for the relation $p(y_1)$. To this end, we calculate the quantity p from Eq. (8) by substituting a certain value of y_1 in it. Setting an initial value $y_1 = 0.0001$ and an increment step $h_1 = 0.001$, we obtain 1370 pairs (y_1, p). Plotting a curve of $p(y_1)$ (Fig. 5) and comparing it with the plot of the power function from [7, p. 579], we see that the formula $p = ay_1^b$ suits to this case. The similarity of the plots can be checked using the smoothening method. In this case, the quantities $X = \ln y_1$ and $Y = \ln p$ are "smoothed": $Y = \ln a + bX$. Calculating X and Y for the given values of y_1 and p, respectively, we infer that the relation between X and Y is almost linear (Fig. 6). From this it follows that the formula is chosen properly. To determine the constants a and b, we construct a linear relation between $\ln y_1$ and $\ln p$ using the following method. We divide

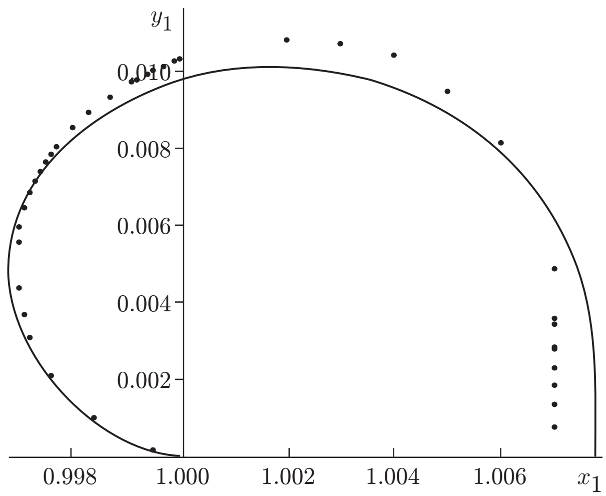

Fig. 1

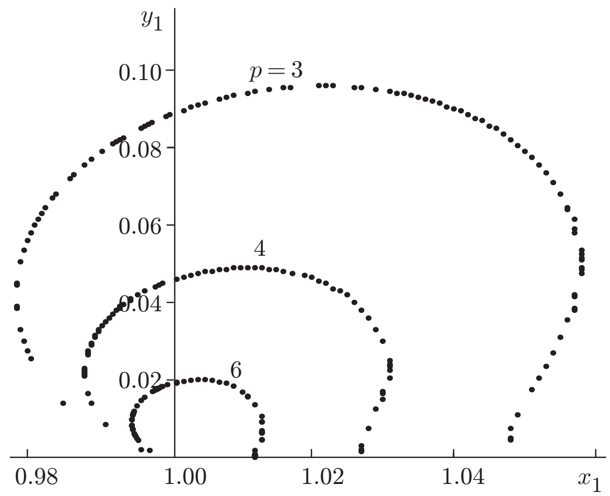

Fig. 2

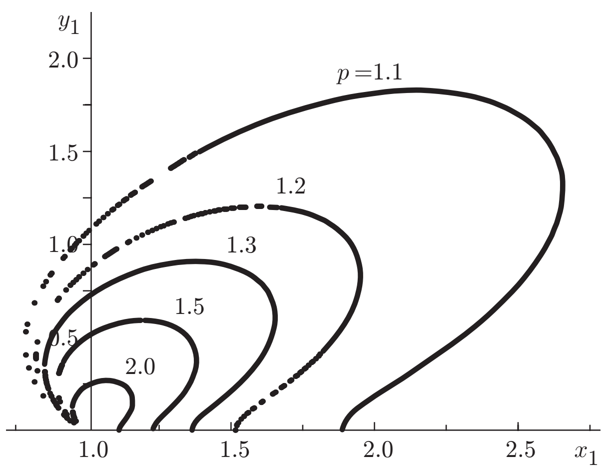

Fig. 3

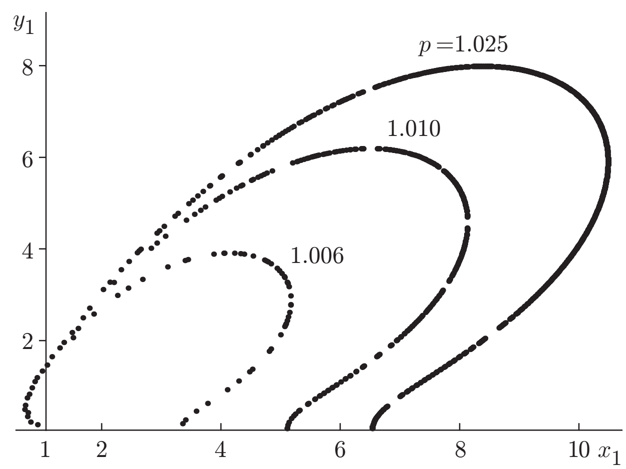

Fig. 4

the conditional equations $Y = \ln a + bX$ for the available pairs of $X_{\rm I}$ and $Y_{\rm I}$ into two groups each containing 685 equations in increasing order of the variable $X_{\rm I}$. Summing up the equations of each group, we obtain the following two equations for a and b:

$$471.4 = 685 \ln a - 948.11b; \qquad 82.65 = 685 \ln a + 5.174b.$$

Hence, $\ln a = 0.1237$, $a = 1.132$, and $b = -0.408$. The values of p are calculated by the formula $p = 1.132 y_1^{-0.408}$. Expressing y_1 in terms of p, we obtain the plastic-zone width:

$$h = y_1 = 1.355 p^{-2.45}.$$

The critical crack-opening displacement is given by

$$h_m = 2 l_0 y(\varepsilon_m - \varepsilon_0) \approx 2.71 p^{-2.45} l_0(\varepsilon_m - \varepsilon_0), \qquad (10)$$

where $\varepsilon_m - \varepsilon_0$ is the maximum specific elongation of the ductile material. For comparison, we give the critical crack-opening displacement obtained by the approximate relation (9) in [2]: $h_m^* = 1.25 p^{-2} l_0(\varepsilon_m - \varepsilon_0)$. The equality $h_m = h_m^*$ holds for $p \approx 5.6$. Thus, the approximate formula (9) can be used only for a limited number of parameters p close to $p \approx 5.6$ (see Table 1).

TABLE 1

p	$h_m/(l_0(\varepsilon_m-\varepsilon_0))$	$h_m^*/(l_0(\varepsilon_m-\varepsilon_0))$
1,006	2.670	1.235
1.2	1.734	0.870
1.5	1.000	0.550
2.0	0.500	0.310
4.0	0.090	0.078
6.0	0.034	0.034
10.0	0.010	0.013
20.0	0.002	0.003

TABLE 2

p	$2\Delta/l_0$	$2\Delta^*/l_0$	Δ^{**}/l_0
1.006	13.08	0.99	105.740
1.010	10.22	0.98	63.300
1.100	1.78	0.83	6.030
1.200	1.03	0.69	2.860
1.500	0.45	0.44	1.000
3.000	0.09	0.11	0.154
10.000	0.01	0.01	0.013

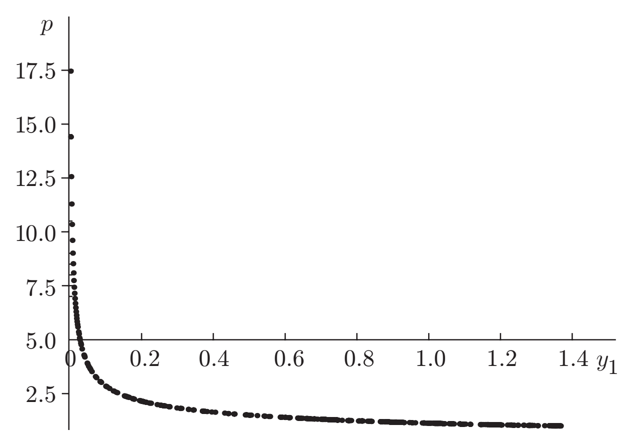

Fig. 5

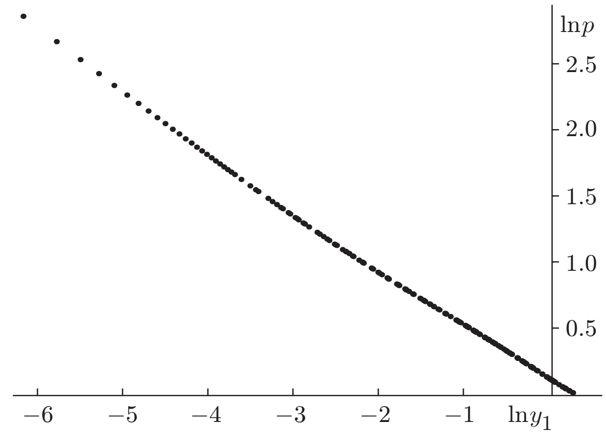

Fig. 6

Let us now estimate the longitudinal dimension of the plastic zone Δ. Setting $y_1 = 0$, we obtain $\theta = \theta_1 = \theta_2 = 0$, $\rho/l_0 = x_1 - 1$, $\rho_1/l_0 = x_1$, and $\rho_2/l_0 = x_1 + 1$. Relations (4), (7), and (8) can be combined to give

$$x_1 = (\sqrt{4p^2-3}+1)/\sqrt{4p^2-6+2\sqrt{4p^2-3}}.$$

Consequently, the estimate of the plastic-zone length obtained by the refined solution has the form

$$\Delta = (x_1-1)l_0 = \Big((\sqrt{4p^2-3}+1)/\sqrt{4p^2-6+2\sqrt{4p^2-3}}-1\Big)l_0. \tag{11}$$

The estimate for the plastic-zone length obtained by the approximate equation (9) is given by [1]

$$\Delta^* = 0.5p^{-2}l_0. \tag{12}$$

Irwin calculated the plastic-zone length Δ^* for the mode I crack assuming, as a first approximation, that the normal stress acting on a segment of length Δ^* is equal to the yield stress of the material [8]. However, when the peak stress is "truncated" by introducing the plastic zone, the equilibrium of the forces exerted by these stresses is disturbed [8], resulting in an underestimated value of Δ^* compared to the real dimensions of the plastic zone. The equilibrium can be attained only by shifting the stress field by a length equal toΔ^*. This procedure reduces to a fictitious increase in the crack length by the quantity Δ^* — the Irwin correction for plasticity. Thus, in the second approximation, where equilibrium of the loads is taken into account, the dimensions of the plastic zone are twice those for the first approximation [8]. We note that formula (11) can be obtained by setting $\sigma_{yy} = \sqrt{\sigma_m^2 - 3\sigma_\infty^2/4} - \sigma_\infty/2$ in the first approximation. Since formulas (10)–(12) were obtained for the first approximation [8], the values of h_m, Δ, and Δ^* should be doubled.

The models that take into account cohesive forces are based on the assumption that cohesive forces exerting resistance to the external loads act on the length Δ^{**} in the vicinity of the crack tip. The cohesive-force intensity depends on the model [9]. The Leonov–Panasyuk–Dugdale model [5, 6], in which the cohesive forces are uniform and equal to the yield stress, is used more frequently than others. The relation

$$\Delta^{**} = l_0[\sec(\pi/(2p)) - 1] \tag{13}$$

defines the plastic-zone length for plane stresses in accordance with this model [1]. We compare the plastic-zone dimensions given by formulas (11)–(13) for the case of plane stresses. Formulas (11) and (12) define the half-length of the plastic zone; therefore, one should compare the quantities $2\Delta/l_0$, $2\Delta^*/l_0$, and Δ^{**}/l_0. According to Table 2, for $p \geqslant 1.5$ (the quasibrittle and brittle types of fracture), the dimensionless quantities $2\Delta/l_0$ and $2\Delta^*/l_0$. agree well. Moderate disagreement between $2\Delta/l_0$ and $2\Delta^*/l_0$ is observed for $1.1 \leqslant p < 1.5$ (quasibrittle fracture). For $1 < p < 1.1$ (the quasiviscous and viscous types of fracture), agreement between $2\Delta/l_0$ and $2\Delta^*/l_0$ is out of the question. Moreover, for $1 \leqslant p < 3$, the plastic-zone length calculated by the Irwin model differs from that calculated for the Leonov–Panasyuk–Dugdale model. For $p \geqslant 3$, the agreement between the quantities obtained for these models can be considered as satisfactory, taking into account that the models differ substantially. It is worth noting that the real dimensions of the plastic zone are much smaller than the dimensions calculated for the Leonov–Panasyuk–Dugdale model, which is due to the triaxial state of stresses [9].

Thus, using the refined solution of the elastoplastic problem, we estimated the plastic-zone dimensions for the quasiviscous and viscous types of fracture, which cannot be done using the approximate solution. It was shown that to calculate the longitudinal dimension of the plastic zone for quasibrittle fracture, it suffices to use the approximate solution, whereas in calculations of the plastic-zone width, the approximate solution can be employed only for a limited set of parameters p. Given estimates of the plastic zone (length 2Δ and width h), one can refine the recommendations on using the sufficient strength criterion for quasiviscous fracture.

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REFERENCES

1. I. M. Kershtein, V. D. Klyushnikov, E. V. Lomakin, and S. A. Shesterikov, *Fundamentals of Experimental Fracture Mechanics* [in Russian], Izd. Mosk. Univ., Moscow (1989).
2. V. M. Kornev, "Generalized sufficient strength criteria. Description of the pre-fracture zone," *J. Appl. Mech. Tech. Phys.*, **43**, No. 5, 763–871 (2002).
3. C. D. Beachem, "Microscopic fracture processes," in: H. Liebowitz (ed.), *Fracture*, Vol. 1: *Microscopic and Macroscopic Fundamentals*, Academic Press, New York–London (1968), pp. 243–350.
4. K. Hellan, *Introduction to Fracture Mechanics*, McGraw-Hill, New York (1984).
5. M. Ya. Leonov and V. V. Panasyuk, "Growth of very small cracks in solids," *Prikl. Mekh.*, **5**, No. 4, 391–401 (1959).
6. D. S. Dugdale, "Yielding of steel sheets containing slit," *J. Mech. Phys. Solids*, **8**, 100–104 (1960).
7. I. N. Bronshtein and K. A. Semendyaev, *Handbook on Mathematics for Engineers and Students* [in Russian], Nauka, Moscow (1986).
8. M. Shiratori, T. Miyoshi, and H. Matsushita, *Computational Fracture Mechanics*, Jikkyo Shuppan Kaisha, Tokyo (1984).
9. G. Pluvinage, *Mechanique of Elastoplastique de la Rupture*, Cepad, Toulouse (1989).